Self-averaging in random self-interacting polygons

2001 J. Phys. A: Math. Gen. 34 L37

(http://iopscience.iop.org/0305-4470/34/6/102)

J. Phys. A: Math. Gen. **34** (2001) L37–L44 www.iop.org/Journals/ja PII: S0305-4470(01)18833-5

LETTER TO THE EDITOR

Self-averaging in random self-interacting polygons

E J Janse van Rensburg[1], E Orlandini[2], M C Tesi[3] and S G Whittington[4]

[1] Department of Mathematics, York University, Toronto, Canada M3J 1P3
[2] INFM, Dipartimento di Fisica, Università di Padova, Padova, Italy
[3] Dipartimento di Matematica, Università di Bologna, Bologna, Italy
[4] Department of Chemistry, University of Toronto, Toronto, Canada M5S 3H6

Received 10 November 2000, in final form 9 January 2001

Abstract
We prove that a lattice polygon model of a self-interacting random ring copolymer is thermodynamically self-averaging. The proof is quite general and applies to any two-body potential linear in the numbers of contacts of different types.

PACS numbers: 0550, 3620, 8270

Quenched random systems have been studied since the early work of Brout (1959) and there has been a resurgence of interest in recent years. In particular there have been several treatments of self-interacting random copolymers (Sfatos and Shakhnovich 1997, Garel *et al* 1998, Monari *et al* 1999, Orlandini *et al* 2000). These polymers could, for instance, have been produced by a polymerization process giving rise to a random sequence of (say) two comonomers A and B. The sequence is random but is then fixed in each individual molecule, so that different molecules have different (fixed) comonomer sequences. Typically one is interested in the situation in which the interaction between a pair of monomers depends on which monomers are involved, so that there are different A–A, B–B and A–B interaction potentials. Under some circumstances such systems can show collapse transitions (see e.g. Monari *et al* 1999).

Although copolymers with different sequences of monomers can show different properties one expects that, in the limit of very long copolymers, the distribution of some properties will be very tightly peaked about their mean values. This is the phenomenon of *self-averaging*. It implies that 'most' (in a sense which can be made precise) sequences of comonomers will give rise to the same properties for the copolymer. If the limiting free energy is independent of the monomer sequence for almost all monomer sequences we say that the system is *thermodynamically self-averaging*.

Thermodynamic self-averaging has been proved rigorously for some spin models (van Hemmen and Palmer 1982, van Enter and van Hemmen 1983), though it is known that correlation functions are not self-averaging in some random spin problems (Derrida and Hilhorst 1981, Sourlas 1987). Thermodynamic self-averaging has also been proved for a self-avoiding walk model of the adsorption of linear copolymers (Orlandini *et al* 1999) and for a lattice tree model of the adsorption of branched copolymers (You and Janse van Rensburg 2000). For self-interacting random copolymers, thermodynamic self-averaging has been proved for an *unfolded* self-avoiding walk model (Orlandini *et al* 2000) but there is no proof available for either a self-avoiding walk model or for a polygon model. In this paper we supply a proof of thermodynamic self-averaging for the polygon case.

The way in which the existence of the thermodynamic limit was proved for the homopolymer version of this problem (i.e. where there is no sequence randomness) was to derive a super-additive inequality for the logarithm of the partition function by a concatenation argument. One might hope to derive a similar relation in the case where the colouring of the vertices is random, to give a super-additive stochastic process, and then use a variant of Kingman's ergodic theorem (Kingman 1968, 1973, Akcoglu and Krengel 1981, Krengel 1985). Indeed this is how self-averaging was proved for the case of unfolded walks (Orlandini *et al* 2000). For the polygon problem things are less simple. As we shall see, it is possible to devise a concatenation construction but the colourings of the polygons are 'inter-leaved' in the construction, so that the results of Kingman and of Akcoglu and Krengel cannot be used directly. In fact the proof that we shall give is self-contained and does not make direct use of results from ergodic theory. The general idea is to concatenate a set of polygons to form a larger polygon and derive an inequality to which one can then apply the strong law of large numbers. The natural way to carry out the concatenation of a set of polygons with given colourings does not always lead to a polygon with the same colouring, because of the inter-leaving of the colours which occurs in the concatenation process. The main difficulty in the proof is to find a scheme to avoid this problem. The situation is somewhat different from the problem of self-averaging in adsorbing lattice trees (Janse van Rensburg 2000, You and Janse van Rensburg 2000), though there are some analogous aspects. The key to the specific solution to the polygon problem is contained in lemma 3, and the later arguments are more general and apply to a wide range of problems.

We consider abstract polygons with n vertices, with vertices independently coloured +1 with probability p and -1 with probability $1-p$. Since the polygon is not rooted, associated with each colouring χ (i.e. with each sequence of +1 and -1) there is a set $\{\chi_j\}$ of cyclic permutations of this colouring, and all members of the set of cyclic permutations must be regarded as one single colouring of the polygon. The polygons are embedded in the d-dimensional hypercubic lattice Z^d, so that each vertex of the polygon has integer coordinates. The vertices are coloured $i = 1, 2, \ldots, n$ and vertex i has coordinates $(x_i, y_i, \ldots, z_i)$. Vertices i and $i+1$ $(i = 1, \ldots, n-1)$ are unit distance apart and are incident on a common edge. Similarly vertices 1 and n are unit distance apart and are incident on a common edge. The bottom (top) vertex of the (embedded) polygon is the vertex with lexicographically first (last) coordinates. A *contact* is a pair of vertices of the polygon which are unit distance apart which are not incident on a common edge of the polygon. Given a colouring χ and a particular cyclic permutation χ_j of χ, let $p_n(k, \chi_j)$ be the number of n-edge polygons, modulo translation, coloured cyclically according to the colouring χ_j, starting at the bottom vertex, with k_{++} contacts between pairs of vertices coloured +1 and +1, k_{--} contacts between pairs of vertices coloured -1 and -1, and k_{+-} contacts between pairs of vertices coloured +1 and -1. We write k for the vector (k_{++}, k_{--}, k_{+-}), and we call k the *contact vector*. Define the corresponding partition function

$$Z_n(\beta, \chi_j) = \sum_k p_n(k, \chi_j) \mathrm{e}^{\beta g(k)} \tag{1}$$

where $g(k)$ is a linear function of the elements of the vector k, bounded above by a constant (γ) multiplied by the number of contacts, and $\beta < \infty$. Since each of the n cyclic permutations of the polygon must be considered we define

$$Z_n^o(\beta,\chi)=\sum_{j=1}^{n} Z_n(\beta,\chi_j) \tag{2}$$

and our aim is show that the free energy

$$\kappa_o(\beta,\chi)=\lim_{n\to\infty} n^{-1}\log Z_n^o(\beta,\chi) \tag{3}$$

exists and is equal, almost surely, to the quenched average free energy

$$\bar{\kappa}(\beta)=\lim_{n\to\infty}\langle n^{-1}\log Z_n^o(\beta,\chi)\rangle \tag{4}$$

where the angular brackets $\langle\cdots\rangle$ denote averaging with respect to the colouring χ.

The *leftmost plane* (*rightmost plane*) of a particular polygon is the plane $x=\min_j\{x_j\}$ ($x=\max_j\{x_j\}$). Consider the set of edges of a polygon which are in the leftmost (rightmost) plane of the polygon, and define the *bottom edge* (*top edge*) as the edge in this set which has lexicographically the least (largest) mid-point. We consider a simpler class of polygons, which we call $*$-*polygons*. A polygon is a $*$-*polygon* if the polygon has a single edge in each of its left- and rightmost planes.

For each polygon there are two paths joining the bottom and top vertices. For fixed α, $0<\alpha<1$ we write $p_n^*(\lfloor\alpha n\rfloor,k,\chi_j)$ for the number of $*$-polygons with n edges, $\lfloor\alpha n\rfloor$ edges in one of the two paths joining the bottom and top vertices, colouring χ_j and contact vector k. We define the two partition functions

$$P_n^*(\lfloor\alpha n\rfloor,\beta,\chi_j)=\sum_k p_n^*(\lfloor\alpha n\rfloor,k,\chi_j)\mathrm{e}^{\beta g(k)} \tag{5}$$

and

$$P_n^*(\beta,\chi_j)=\int \mathrm{d}\alpha\, P_n^*(\lfloor\alpha n\rfloor,\beta,\chi_j). \tag{6}$$

We next prove a series of lemmas about $*$-polygons.

Lemma 1. *The quenched average free energy*

$$\kappa^*(\alpha,\beta)=\lim_{n\to\infty}\langle n^{-1}\log P_n^*(\lfloor\alpha n\rfloor,\beta,\chi_j)\rangle \tag{7}$$

exists for all $\beta<\infty$.

Proof. For any $\beta<\infty$, $n^{-1}\log P_n^*(\lfloor\alpha n\rfloor,\beta,\chi_j)$ is bounded above since the total number of polygons increases exponentially and $g(k)\leqslant\gamma(d-1)n$. Let Q_1 be a polygon counted by $p_n^*(\lfloor\alpha n\rfloor,k-k_1,\chi_1)$ and Q_2 be a polygon counted by $p_m^*(\lfloor\alpha m\rfloor,k_1,\chi_2)$. Translate Q_2 such that the midpoint of its bottom edge has first coordinate one larger than the first coordinate of the midpoint of the top edge of Q_1. Rotate Q_2 about the first direction until its bottom edge is parallel to the top edge of Q_1. Notice that there are no nearest-neighbour contacts between vertices in Q_1 and Q_2, except for endpoints of the top and bottom edges. If these top and bottom edges are deleted and two edges are added to join Q_1 and Q_2 to form a single polygon, then two new contacts are created. The rotation of Q_2 can be carried out such that the resulting polygon has two disjoint paths between the bottom and top vertices, of lengths $\lfloor\alpha n\rfloor+\lfloor\alpha m\rfloor$ and $(n+m)-(\lfloor\alpha n\rfloor+\lfloor\alpha m\rfloor)$. Let l be the contact vector of the new polygon, and define $k_0=l-k$. The colouring of the new polygon is that sequence of colours, χ_3, obtained via the concatenation construction. The colouring χ_1 of Q_1 can be regarded as a set of two colourings, one (χ_1^α, say) of length $\lfloor\alpha n\rfloor$ and a second ($\chi_1^{1-\alpha}$, say) of length $n-\lfloor\alpha n\rfloor$. Similarly the colouring χ_2 of Q_2 can be regarded as a set of two colourings, one (χ_2^α, say) of length $\lfloor \alpha m \rfloor$ and a second ($\chi_2^{1-\alpha}$, say) of length $m - \lfloor \alpha m \rfloor$. Then the colouring χ_3 is the concatenation $\chi_1^\alpha + \chi_2^\alpha + \chi_2^{1-\alpha} + \chi_1^{1-\alpha}$. Observe that the same χ_3 is obtained for any Q_1 and Q_2, given χ_1 and χ_2. Similarly, k_0 is also fixed, depending only on α, and χ_1 and χ_2. This gives the supermultiplicative inequality

$$\sum_{k_1} p_n^*(\lfloor \alpha n \rfloor, k - k_1, \chi_1) p_m^*(\lfloor \alpha m \rfloor, k_1, \chi_2) \leqslant 2(d-1) p_{n+m}^*(\lfloor \alpha n \rfloor + \lfloor \alpha m \rfloor, k + k_0, \chi_3) \qquad (8)$$

since at most $2(d-1)$ copies of Q_2 could be rotated to become identical. Since two contacts are created, the elements of the contact vector k_0 are each at most 2 and at least 0.

Multiply both sides by $e^{\beta g(k)} = e^{\beta g(k-k_1)} e^{\beta g(k_1)}$ (since g is a linear function) and sum over k. If we define $h(\beta) = \max_{k_0}\{\beta g(k_0)\}$ (now independent of Q_1 and Q_2), then the sums over k and k_1 can be performed to give

$$P_n^*(\lfloor \alpha n \rfloor, \beta, \chi_1) P_m^*(\lfloor \alpha m \rfloor, \beta, \chi_2) \leqslant 2(d-1) e^{h(\beta)} P_{m+n}^*(\lfloor \alpha n \rfloor + \lfloor \alpha m \rfloor, \beta, \chi_3). \qquad (9)$$

Take logarithms and then average over χ_1 and χ_2 to obtain

$$\begin{aligned}\langle \log P_n^*(\lfloor \alpha n \rfloor, \beta, \chi)\rangle_\chi + \langle \log P_m^*(\lfloor \alpha m \rfloor, \beta, \chi)\rangle_\chi &\leqslant h(\beta) + \log(2(d-1)) \\ &\quad + \langle \log P_{m+n}^*(\lfloor \alpha n \rfloor + \lfloor \alpha m \rfloor, \beta, \chi)\rangle_\chi. \end{aligned} \qquad (10)$$

Equation (10) is a generalized super-additive inequality and the existence of the limit is a consequence of theorem 3.4 in Janse van Rensburg (2000). □

Note that, by symmetry,

$$p_n^*(\lfloor \alpha n \rfloor, k, \chi_j) = p_n^*(\lfloor (1-\alpha) n \rfloor, k, \bar{\chi}_j) \qquad (11)$$

where $\bar{\chi}_j$ is χ_j read in reverse order. Thus, it follows that

$$\kappa^*(\alpha, \beta) = \kappa^*(1-\alpha, \beta). \qquad (12)$$

Lemma 2. *The free energy $\kappa^*(\alpha, \beta)$ is a concave function of α and*

$$\kappa^*(\alpha, \beta) \leqslant \kappa^*(1/2, \beta) \qquad (13)$$

for every fixed $\beta < \infty$.

Proof. By concatenating two polygons, each with n edges, as in the proof of lemma 1, one can obtain the inequality

$$P_n^*(\lfloor \alpha_1 n \rfloor, \beta, \chi_1) P_n^*(\lfloor \alpha_2 n \rfloor, \beta, \chi_2) \leqslant 2(d-1) e^{h(\beta)} P_{2n}^*(\lfloor \alpha_1 n \rfloor + \lfloor \alpha_2 n \rfloor, \beta, \chi_3). \qquad (14)$$

Replacing α_1 by α, and α_2 by $1-\alpha$, taking logarithms, dividing by n, averaging over χ_1 and χ_2 and taking $n \to \infty$ gives

$$\kappa^*(\alpha, \beta) + \kappa^*(1-\alpha, \beta) \leqslant 2\kappa^*(1/2, \beta). \qquad (15)$$

The concavity of κ^* in α now follows since $\kappa^*(\alpha, \beta)$ is a monotonic, bounded function of α at fixed $\beta < \infty$. Using the concavity of $\kappa^*(\alpha, \beta)$ together with equation (11) gives (13). □

We next show self-averaging for the free energy of $*$-polygons for any fixed α and β. The proof is similar in spirit to lemma 6.41 and theorem 6.42 in Janse van Rensburg (2000), and consists of several parts.

Lemma 3. *For fixed $\beta < \infty$, and for any α and almost all fixed colourings χ_0 we have*

$$\kappa^*(\alpha, \beta) \leqslant \liminf_{n \to \infty} n^{-1} \log P_n^*(\lfloor \alpha n \rfloor, \beta, \chi_0). \qquad (16)$$

Proof. Let $n = pm + q$ for fixed m, where $0 \leqslant q < m$. Concatenate p polygons each of size m, and one polygon of size q to form a polygon of size n. Choose the colouring of the ith polygon to be χ_i, and that of the final polygon to be χ_{p+1}, such that the concatenated polygon of length n has colouring χ_0. If all these polygons have fixed α, then the resulting colouring χ_0 is independent of the particular embeddings of the polygons. This results in the inequality

$$P^*_{pm+q}(p\lfloor \alpha m\rfloor + \lfloor \alpha q\rfloor, \beta, \chi_0) \geqslant \left[\prod_{i=1}^{p}(2(d-1)\,\mathrm{e}^{h(\beta)} P^*_m(\lfloor \alpha m\rfloor, \beta, \chi_i))\right] P^*_q(\lfloor \alpha q\rfloor, \beta, \chi_{p+1}). \tag{17}$$

Take logarithms and dividing by $mp + q$ gives

$$\frac{1}{pm+q}\log P^*_{pm+q}(p\lfloor \alpha m\rfloor + \lfloor \alpha q\rfloor, \beta, \chi_0) \geqslant \frac{pm}{pm+q}\left[\frac{1}{p}\sum_{i=1}^{p}\left(\frac{1}{m}\log P^*_m(\lfloor \alpha m\rfloor, \beta, \chi_i)\right)\right] + \frac{p[h(\beta) + \log(2(d-1))]}{mp+q} + \frac{1}{mp+q}\log P^*_q(\lfloor \alpha q\rfloor, \beta, \chi_{p+1}). \tag{18}$$

If $p \to \infty$ with fixed m, then by the strong law of large numbers if follows that

$$\liminf_{n\to\infty} n^{-1}\log P^*_n(\lfloor \alpha n\rfloor, \beta, \chi_0) \geqslant \langle m^{-1}\log P^*_m(\lfloor \alpha m\rfloor, \beta, \chi)\rangle_\chi + \frac{h(\beta) + \log(2(d-1))}{m} \tag{19}$$

for almost all colourings χ_0. If $m \to \infty$, then the lemma follows from lemma 1. □

Lemma 4. *For almost all colourings* χ_0

$$\kappa^*(\alpha, \beta) = \liminf_{n\to\infty} n^{-1}\log P^*_n(\lfloor \alpha n\rfloor, \beta, \chi_0). \tag{20}$$

Proof. Since the space Ω of all sequences of colours is a probability space with uniform measure, it follows that

$$\kappa^*(\alpha, \beta) = \lim_{n\to\infty}\int_\Omega \mathrm{d}\chi\,[n^{-1}\log P^*_n(\lfloor \alpha n\rfloor, \beta, \chi)] \tag{21}$$

and from Fatou's lemma (see for instance Friedman 1982) we obtain

$$\kappa^*(\alpha, \beta) \geqslant \int_\Omega \mathrm{d}\chi\,\liminf_{n\to\infty} n^{-1}\log P^*_n(\lfloor \alpha n\rfloor, \beta, \chi). \tag{22}$$

Define the decomposition $\Omega = \Omega_- \cup \Omega_0 \cup \Omega_+$ by

$$\begin{aligned}
&\liminf_{n\to\infty} n^{-1}\log P^*_n(\lfloor \alpha n\rfloor, \beta, \chi) = \kappa^*(\alpha, \beta) && \forall\chi \in \Omega_0\\
&\liminf_{n\to\infty} n^{-1}\log P^*_n(\lfloor \alpha n\rfloor, \beta, \chi) < \kappa^*(\alpha, \beta) && \forall\chi \in \Omega_-\\
&\liminf_{n\to\infty} n^{-1}\log P^*_n(\lfloor \alpha n\rfloor, \beta, \chi) > \kappa^*(\alpha, \beta) && \forall\chi \in \Omega_+.
\end{aligned} \tag{23}$$

By lemma 3 the measure of Ω_- is zero. Suppose that the measure of Ω_+, $\mu(\Omega_+)$, is positive. Then $\mu(\Omega_0) = 1 - \mu(\Omega_+)$. Therefore

$$\int_\Omega \mathrm{d}\chi\,[\liminf_{n\to\infty} n^{-1}\log P^*_n(\lfloor \alpha n\rfloor, \beta, \chi)] > \mu(\Omega_+)\kappa^*(\alpha, \beta) + (1 - \mu(\Omega_+))\kappa^*(\alpha, \beta) = \kappa^*(\alpha, \beta). \tag{24}$$

This contradicts equation (22), and we conclude that $\mu(\Omega_+) = 0$. This proves the lemma. □

Theorem 1. *For almost all colourings* χ_0

$$\kappa^*(\alpha, \beta) = \lim_{n\to\infty} n^{-1} \log P_n^*(\lfloor \alpha n \rfloor, \beta, \chi_0). \tag{25}$$

Proof. Suppose that

$$\limsup_{n\to\infty} n^{-1} \log P_n^*(\lfloor \alpha n \rfloor, \beta, \chi) > \kappa^*(\alpha, \beta) \tag{26}$$

for all $\chi \in U$ where the measure of U, $\mu(U)$, is positive. By definition of the lim sup, there is an $\epsilon_\chi > 0$ and an infinite set of integers $\{n_i\}$ such that $n_i^{-1} \log P_{n_i}^*(\lfloor \alpha n_i \rfloor, \beta, \chi)$ is convergent when $i \to \infty$ and such that for each such n_i,

$$n_i^{-1} \log P_{n_i}^*(\lfloor \alpha n_i \rfloor, \beta, \chi) > \kappa^*(\alpha, \beta) + \epsilon_\chi \tag{27}$$

for each $\chi \in U$. Define the function $T_n = \frac{1}{n_{i+1}} \log P_{n_{i+1}}^*(\lfloor \alpha n_{i+1} \rfloor, \beta, \chi)$ for $n_i < n \leqslant n_{i+1}$. Then for all $n \geqslant n_1$,

$$T_n > \kappa^*(\alpha, \beta) + \epsilon_\chi. \tag{28}$$

T_n is measurable on Ω, so by the Lebesgue dominated convergence theorem it follows that

$$\int_\Omega \mathrm{d}\chi \lim_{n\to\infty} T_n = \lim_{n\to\infty} \int_\Omega \mathrm{d}\chi\, T_n = \lim_{i\to\infty} \int_\Omega \mathrm{d}\chi\, \frac{1}{n_i} \log P_{n_i}^*(\lfloor \alpha n_i \rfloor, \beta, \chi). \tag{29}$$

Thus, from equation (21) it follows that

$$\kappa^*(\alpha, \beta) = \lim_{n\to\infty} \int_\Omega \mathrm{d}\chi\, n^{-1} \log P_n^*(\lfloor \alpha n \rfloor, \beta, \chi) \geqslant \kappa^*(\alpha, \beta) + \int_U \mathrm{d}\chi\, \epsilon_\chi > \kappa^*(\alpha, \beta). \tag{30}$$

This gives a contradiction, unless $\mu(U) = 0$, which proves the theorem. □

This theorem proves self-averaging for every fixed value of α, and in particular for $\alpha = 1/2$. We now show that the model of $*$-polygons is self-averaging by relating $P_n^*(\beta, \chi_j)$ and $P_n^*(n/2, \beta, \chi_j)$.

Theorem 2. *For all* $\beta < \infty$ *and almost all* χ_0

$$\lim_{n\to\infty} n^{-1} \log P_n^*(\beta, \chi_0) = \lim_{n\to\infty} \langle n^{-1} \log P_n^*(\beta, \chi) \rangle_\chi. \tag{31}$$

Proof. By inclusion,

$$\frac{1}{n} \log P_n^*(n/2, \beta, \chi) \leqslant \frac{1}{n} \log P_n^*(\beta, \chi). \tag{32}$$

To obtain a bound in the other direction we observe that

$$\begin{aligned} P_n^*(\beta, \chi) &= \int_0^1 \mathrm{d}\alpha\, P_n^*(\lfloor \alpha n \rfloor, \beta, \chi) \\ &= P_n^*(n/2, \beta, \chi) \int_0^1 \mathrm{d}\alpha\, \mathrm{e}^{\kappa^*(\alpha,\beta)n - \kappa^*(1/2,\beta)n + \mathrm{o}(n)} \\ &\leqslant P_n^*(n/2, \beta, \chi) \mathrm{e}^{\mathrm{o}(n)} \end{aligned} \tag{33}$$

since $\kappa^*(\alpha, \beta) \leqslant \kappa^*(1/2, \beta)$. The squeeze theorem for limits then gives the desired result. □

We next relate $P_n^*(\beta, \chi)$ to $Z_n(\beta, \chi)$. Each polygon can be converted to a $*$-polygon with four additional edges and four additional vertices. These additional vertices can be coloured in 2^4 different ways. Given a colouring χ_0 of the original polygon, there are 2^4 possible colourings of the resulting $*$-polygon which we call χ_j^*, $j = 1, 2, \ldots, 2^4$. Then we have the inequality

$$Z_n(\beta, \chi_0) \leqslant \sum_{j=1}^{2^4} P^*_{n+4}(\beta, \chi_j^*) \leqslant 2^4 P^*_{n+4}(\beta, \chi^*_{j_0}) \tag{34}$$

where β is fixed, and where j_0 is that value of j which maximizes the summand above. Observe that j_0 depends on $g(k)$ and on χ_0. Any polygon counted by $P^*_{n+4}(\beta, \chi^*_{j_0})$ can be converted into a polygon counted by $Z_n(\beta, \chi_0)$ by removing the four vertices in the left- and rightmost planes, and adding the two obvious edges to form a polygon. This deletes two contacts, and the energy change can be bounded above by a constant. Therefore,

$$P^*_{n+4}(\beta, \chi^*_{j_0}) \leqslant A(\beta) Z_n(\beta, \chi_0). \tag{35}$$

Taking logarithms, dividing by n, and letting $n \to \infty$ in equations (34) and (35) gives

$$\lim_{n\to\infty} n^{-1} \log Z_n(\beta, \chi_0) = \kappa^*(1/2, \beta) = \kappa^*(\beta) \tag{36}$$

for almost all χ_0. The result is the following theorem.

Theorem 3. *The lattice polygon model of self-interacting random ring copolymers is thermodynamically self-averaging. That is*

$$\lim_{n\to\infty} n^{-1} \log Z_n(\beta, \chi_0) = \lim_{n\to\infty} \langle n^{-1} \log Z_n(\beta, \chi)\rangle_\chi \tag{37}$$

for almost all χ_0.

Finally we turn to the relation between $Z_n(\beta, \chi_j)$ and $Z_n^o(\beta, \chi)$ where χ_j is one of the n cyclic permutations of the colouring χ. From equation (2)

$$Z_n^o(\beta, \chi) \leqslant n \max_j Z_n(\beta, \chi_j) \equiv n Z_n(\beta, \chi_{j_o}) \tag{38}$$

and

$$Z_n^o(\beta, \chi) \geqslant Z_n(\beta, \chi_{j_o}). \tag{39}$$

Therefore

$$\kappa_o(\beta, \chi) \equiv \lim_{n\to\infty} n^{-1} \log Z_n^o(\beta, \chi) = \kappa^*(\beta) \tag{40}$$

for almost all colourings χ, and is equal to the quenched average free energy $\bar{\kappa}(\beta)$.

We wish to emphasize that the model for which we have proved thermodynamic self-averaging is an important physical model of self-interacting random copolymers. However, the problem of self-averaging in the corresponding self-avoiding walk model is still open.

Some of this work was carried out while three of the authors were visiting the Mathematisches Forschunginstitut Oberwolfach. This research was financially supported, in part, by NSERC of Canada.

References

Akcoglu M A and Krengel U 1981 *J. Reine Angew. Math.* **323** 53–67
Brout R 1959 *Phys. Rev.* **115** 824–35
Derrida B and Hilhorst H 1981 *J. Phys. C: Solid State Phys.* **14** L539–44
Friedman A 1982 *Foundations of Modern Analysis* (New York: Dover)
Garel T, Orland H and Pitard E 1998 Protein folding and heteropolymers *Spin Glasses and Random Fields* ed A P Young (Singapore: World Scientific) pp 387–443

Janse van Rensburg E J 2000 *The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles (Oxford Lecture Series in Math and its Applications vol 18)* ed J Ball and D Welsh (Oxford: Oxford University Press)
Kingman J F C 1968 *J. R. Stat. Soc.* B **30** 499–510
——1973 *Ann. Prob.* **1** 883–909
Krengel U 1985 *Ergodic Theorems* (Berlin: de Gruyter)
Monari P, Stella A L, Vanderzande C and Orlandini E 1999 *Phys. Rev. Lett.* **83** 112–6
Orlandini E, Tesi M C and Whittington S G 1999 *J. Phys. A: Math. Gen.* **32** 469–77
——2000 *J. Phys. A: Math. Gen.* **33** 259–66
Sfatos C D and Shakhnovich E I 1997 *Phys. Rep.* **288** 77–108
Sourlas N 1987 *Europhys. Lett.* **3** 1007–11
van Enter A C D and van Hemmen J L 1983 *J. Stat. Phys.* 141–52
van Hemmen J L and Palmer R G 1982 *J. Phys. A: Math. Gen.* **15** 3881–90
You S and Janse van Rensburg E J 2000 *J. Phys. A: Math. Gen.* **33** 1171–86